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Theoretical study of the slow motion of a sphere and a fluid in a cylindrical tube

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The results of a theoretical treatment are presented for the slow flow of a viscous fluid through a circular cylinder within which a small spherical particle is confined. The sphere is situated in an arbitrary position within the cylinder, rotates with an arbitrary constant angular velocity and moves at constant velocity parallel to the wall. Approximate expressions are presented which give the frictional force, torque, and permanent pressure drop caused by the presence of this obstacle in the original Poiseuillian field of flow.

An eccentricity function for the torque on a sphere in a circular cylinder was evaluated numerically. It can be used to predict the wall-effect for the torque as well as the angular velocity with which a 'dense' spherical particle will rotate. Expressions are presented which predict the angular velocity of 'dense' as well as neutrally buoyant hydrodynamically supported spherical particles.

1. Introduction

The behaviour of particles of approximately spherical shape suspended in fluids is of fundamental importance in problems involving sedimentation, packed beds, fluidization, hindered settling and suspension viscosity. As a preliminary to understanding the behaviour of multiparticle systems, we consider here the dynamics of a single particle.

Employing the method of reflexions, Brenner & Happel (1958) and Happel & Brenner (1957, 1965) considered the case of the slow translation of a single spherical particle which is kept from rotating as it moves parallel to the longitudinal axis of an infinitely long circular cylinder through which a viscous fluid may be flowing. Expressions for the frictional force, torque, and pressure drop are developed which are accurate for small a/R_0 (first-order corrections). We shall extend the problem treated by Brenner & Happel (1958) to the more general case where the sphere may also rotate with an arbitrary constant angular velocity as it slowly translates parallel to the longitudinal axis of an infinitely long circular cylinder. Second-order corrections will be developed by utilizing the entire velocity field reflected from the sphere.

Happel & Brenner (1965) have recently reviewed the literature on this subject. Detailed derivation of the expressions appearing in this paper are presented in Greenstein's (1967) thesis.

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2. Description of the problem

Let us consider the slow translation and rotation of a spherical particle moving with an arbitrary constant translational velocity and rotating with an arbitrary constant angular velocity through a viscous incompressible fluid confined within an infinitely long circular cylindrical tube. The sphere moves with an arbitrary constant translational velocity, $\mathbf{U}$, relative to the cylinder wall in the direction

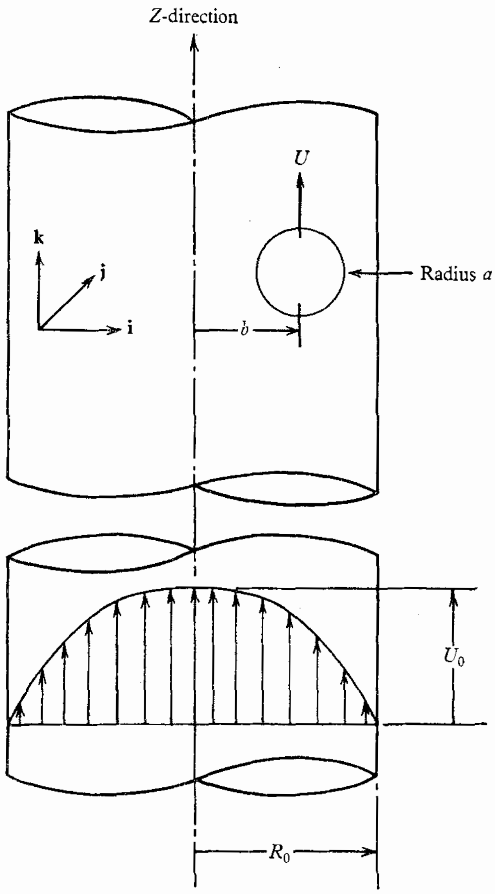

FIGURE 1. Motion of sphere and fluid in a circular cylindrical tube.

of Z positive, parallel to the cylinder axis, and rotates with an arbitrary constant angular velocity, $\mathbf{\Omega} = \mathbf{i}\Omega_1 + \mathbf{j}\Omega_2 + \mathbf{k}\Omega_3$, relative to the cylinder wall, while the fluid flows in laminar flow with a superficial velocity of $\frac{1}{2}U_0$ in the positive Z-direction. The sphere radius is a, the cylinder radius is R_0, and the centre of the sphere is situated at a distance b from the cylinder axis in the $\mathbf{i}$-direction, as shown in figure 1.

It is assumed that the fluid motion is governed by the creeping motion and continuity equations

$$\nabla^2\mathbf{V} = \frac{1}{\mu}\mathbf{\nabla}p, \quad \mathbf{\nabla}.\mathbf{V} = 0. \qquad (2.1, 2.2)$$

V is the fluid velocity with respect to a co-ordinate system which moves with the sphere, p is the dynamic pressure and μ the fluid viscosity. In terms of a co-ordinate system which moves with the sphere the boundary conditions which define the fluid velocity field **V** are: (i) at fluid–solid interfaces there is no relative motion; (ii) at large distances from the disturbing influence of the sphere, $z = \pm\infty$, the velocity distribution becomes Poiseuillian.

The above boundary-value problem can be solved by a technique of successive approximations known as the method of reflexions (see Happel & Brenner 1965). Since the equations of motion and boundary conditions are linear, the frictional force **F** and torque **T** (about the sphere centre) exerted on the sphere by the fluid, and the additional pressure drop ΔP_S (above that due to the original Poiseuillian field, ΔP_0) experienced by the fluid during its passage through the cylinder as a result of the presence of the sphere, may be obtained by summing the respective contributions of each of the individual fields:

$$\mathbf{F} = \sum_{i=0}^{\infty} \mathbf{F}^{(i)}, \tag{2.3}$$

$$\mathbf{T} = \sum_{i=0}^{\infty} \mathbf{T}^{(i)}, \tag{2.4}$$

$$\Delta P_S = \sum_{i=1}^{\infty} \Delta P_i. \tag{2.5}$$

3. Final results for off-centre sphere

If the results for each of the individual six fields are added in accordance with (2.3)–(2.5) we obtain the following expressions for the frictional force and torque experienced by the sphere and the pressure drop experienced by the fluid as a result of the presence of the sphere. For brevity we have set $\beta = b/R_0$.

$$\left.\begin{aligned} \mathbf{F} &= \mathbf{F}^{(1)} + \mathbf{F}^{(2)} + \mathbf{F}^{(3)} + \mathbf{F}^{(4)} + \mathbf{F}^{(5)} + \mathbf{F}^{(6)} + \dots \\ &= -\mathbf{k}6\pi\mu a\left\{(U - U_0(1-\beta^2))\left[1 + f(\beta)\left(\frac{a}{R_0}\right) + f^2(\beta)\left(\frac{a}{R_0}\right)^2\right]\right. \\ &\quad \left. + \tfrac{2}{3}U_0\left(\frac{a}{R_0}\right)^2\right\} + O\left(\frac{a}{R_0}\right)^3. \end{aligned}\right\} \tag{3.1}$$

$$\left.\begin{aligned} \mathbf{T} &= \mathbf{T}^{(1)} + \mathbf{T}^{(2)} + \mathbf{T}^{(3)} + \mathbf{T}^{(4)} + \mathbf{T}^{(5)} + \mathbf{T}^{(6)} + \dots \\ &= -\mathbf{i}8\pi\mu a^3\Omega_1 \\ &\quad -\mathbf{j}\left\{8\pi\mu a^3\left(\Omega_2 - \frac{U_0 b}{R_0^2}\right) + 8\pi\mu a^2[U - U_0(1-\beta^2)]g(\beta)\left(\frac{a}{R_0}\right)^2\left[1 + g(\beta)\left(\frac{a}{R_0}\right)\right]\right\} \\ &\quad - \mathbf{k}8\pi\mu a^3\Omega_3 + O\left(\frac{a}{R_0}\right)^4. \end{aligned}\right\} \tag{3.2}$$

$$\left.\begin{aligned}\Delta P_S &= (\Delta P_1+\Delta P_2)+(\Delta P_3+\Delta P_4)+(\Delta P_5+\Delta P_6)+\ldots\\ &= -\frac{12\mu a}{R_0^2}\left\{U_0\left[(1-\beta^2)^2+\left(\frac{44}{9}\beta^2-\frac{4}{3}\right)\left(\frac{a}{R_0}\right)^2\right]-U\left[(1-\beta^2)-\frac{2}{3}\left(\frac{a}{R_0}\right)^2\right]\right.\\ &\quad\left.+[U_0(1-\beta^2)-U](1-\beta^2)f(\beta)\left(\frac{a}{R_0}\right)\left[1+f(\beta)\left(\frac{a}{R_0}\right)\right]-\frac{4}{3}\Omega_2 b\left(\frac{a}{R_0}\right)^2+O\left(\frac{a}{R_0}\right)^3\right\}.\end{aligned}\right\} \tag{3.3}$$

A form of the preceding equations suitable for examining situations in which the sphere is near the container walls can be obtained by expressing the previous results in terms of the ratio of the sphere radius to minimum distance (of the

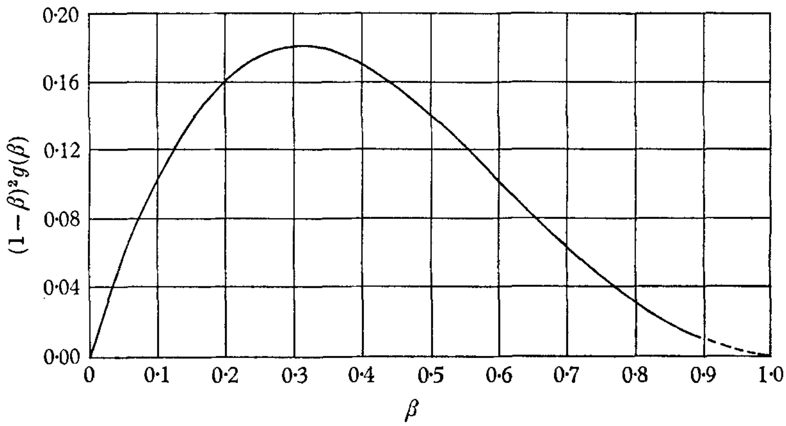

FIGURE 2. Eccentricity function for the torque on a sphere in a circular cylinder.

sphere centre) from the wall, $a/(R_0-b)$. This is done by replacing a/R_0 by $[1-(b/R_0)][a/(R_0-b)]$. Although we can continue the reflexion process indefinitely, it is meaningless to continue beyond the sixth reflexion since the contribution of succeeding terms is of the same or larger order than those which we have neglected in the approximation.

The functions $f(\beta)$ and $g(\beta)$ have been previously defined (Brenner & Happel 1958). The function $f(\beta)$ has been evaluated numerically by Famularo (1962) and the results tabulated in table 1 of his thesis (pp. 62–3) and in Happel & Brenner (1965, p. 309). In order to facilitate interpolation Greenstein (1967) recalculated the function $f(\beta)$. Values of $f(\beta)$ and $(1-\beta)f(\beta)$ *vs.* β are presented in table 1; additional values of $f(\beta)$ appear here which have not been reported previously. Furthermore, the values of $f(\beta)$ when $\beta \geqslant 0{\cdot}60$ reported previously are not correct to the number of significant figures previously reported.

The function $g(\beta)$ has been evaluated numerically, by Greenstein (1967), for various values of the parameter (β), and the results obtained are tabulated in table 2. A plot of the function $(1-\beta)^2 g(\beta)$ *vs.* β is presented in figure 2.

Various particular cases may be derived from our work by making the following substitutions. When the sphere is kept from rotating, set $\Omega_1 = \Omega_2 = \Omega_3 = 0$. When the sphere is kept from translating, set $U = 0$. When the fluid is quiescent, set $U_0 = 0$.

For a freely suspended sphere the resultant of the upward frictional force and downward gravity force is zero, i.e.

$$\mathbf{F} = \mathbf{k}W. \tag{3.4}$$

Now,

$$W = \tfrac{4}{3}\pi a^3(\rho_S - \rho)g, \tag{3.5}$$

β	$f(\beta)$	$(1-\beta)f(\beta)$	β	$f(\beta)$	$(1-\beta)f(\beta)$
0·00	2·10444	2·10444	0·40	2·04388	1·22633
0·01	2·10433	2·08329	0·41	2·04391	1·20591
0·02	2·10415	2·06207	0·43	2·04522	1·16577
0·03	2·10381	2·04069	0·45	2·04819	1·12651
0·05	2·10270	1·99756	0·50	2·06557	1·03278
0·10	2·09758	1·88783	0·55	2·10274	0·946233
0·15	2·08962	1·77618	0·60	2·16980	0·867920
0·20	2·07937	1·66350	0·65	2·28060	0·798210
0·25	2·06801	1·55101	0·70	2·45850	0·737550
0·30	2·05687	1·43981	0·75	2·742	0·6855
0·35	2·04800	1·33120	0·80	3·20	0·640
0·37	2·04561	1·28874	0·85	3·96	0·594
0·39	2·04419	1·24695	0·90	5·30	0·530

TABLE 1. Tabulation of $f(\beta)$ and $(1-\beta)f(\beta)$ *vs.* β

β	$g(\beta)$	$(1-\beta)^2g(\beta)$	β	$g(\beta)$	$(1-\beta)^2g(\beta)$
0·00	0	0	0·32	0·393691	0·182042
0·01	0·0129614	0·0127035	0·33	0·404624	0·181635
0·02	0·0259183	0·0248919	0·35	0·426101	0·180027
0·03	0·0388690	0·0365718	0·40	0·477443	0·171879
0·04	0·0518074	0·0477457	0·45	0·525110	0·158846
0·05	0·0647301	0·0584190	0·50	0·568742	0·142185
0·08	0·1033672	0·0874900	0·55	0·60823	0·12316
0·10	0·128974	0·104469	0·60	0·64376	0·10300
0·15	0·192253	0·138903	0·65	0·67574	0·082778
0·20	0·254081	0·162612	0·70	0·7059	0·06354
0·25	0·313972	0·176609	0·75	0·7378	0·04611
0·27	0·33727	0·17973	0·80	0·7802	0·03121
0·29	0·360192	0·181573	0·85	0·857	0·0193
0·30	0·371474	0·182022	0·90	1·03	0·0103
0·31	0·382645	0·182177			

TABLE 2. Values of eccentricity function $g(\beta)$

where ρ_S and ρ are the densities of particle and fluid, respectively, and g is the magnitude of the local acceleration of gravity. Solving (3.1) for $[U-U_0(1-\beta^2)]$, we obtain

$$\begin{aligned} U-U_0(1-\beta^2) &= \frac{-\dfrac{W}{6\pi\mu a} - \tfrac{2}{3}U_0\left(\dfrac{a}{R_0}\right)^2}{1+f(\beta)\left(\dfrac{a}{R_0}\right)+f^2(\beta)\left(\dfrac{a}{R_0}\right)^2} + O\left(\frac{a}{R_0}\right)^3 \\ &= \frac{-W}{6\pi\mu a}\left[1-f(\beta)\left(\frac{a}{R_0}\right)\right] - \tfrac{2}{3}U_0\left(\frac{a}{R_0}\right)^2 + O\left(\frac{a}{R_0}\right)^3. \end{aligned} \tag{3.6}$$

Since the sphere is free to rotate the resultant torque acting on it should vanish, i.e. $\mathbf{T}_0 = \mathbf{0}$. Setting $\mathbf{T}_0 = \mathbf{0}$ in (3.2) and solving for $a\Omega_1$, $a\Omega_2$ and $a\Omega_3$, we obtain

$$\left.\begin{aligned} a\Omega_1 &= 0, \\ a\Omega_2 &= U_0\frac{a}{R_0}\frac{b}{R_0} - [U - U_0(1-\beta^2)]g(\beta)\left(\frac{a}{R_0}\right)^2\left[1 + g(\beta)\left(\frac{a}{R_0}\right)\right] + O\left(\frac{a}{R_0}\right)^4, \\ a\Omega_3 &= 0. \end{aligned}\right\} \quad (3.7)$$

Substituting the value of $[U - U_0(1-\beta^2)]$, given by (3.6), into the above equation, we obtain

$$a\Omega_2 = U_0\frac{a}{R_0}\frac{b}{R_0} + \frac{W}{6\pi\mu a}g(\beta)\left(\frac{a}{R_0}\right)^2\left[1 + \frac{a}{R_0}(g(\beta) - f(\beta))\right] + O\left(\frac{a}{R_0}\right)^4. \quad (3.8)$$

For a neutrally buoyant particle ($W = 0$) the above equation takes the form

$$a\Omega_2 = \frac{U_0 ab}{R_0^2} + O\left(\frac{a}{R_0}\right)^4, \quad (3.9)$$

while (3.6) reduces to

$$[U - U_0(1-\beta^2)] = -\tfrac{2}{3}U_0\left(\frac{a}{R_0}\right)^2 + O\left(\frac{a}{R_0}\right)^3. \quad (3.10)$$

It has been shown that in the Stokes (low Reynolds number) approximation a freely suspended particle placed in a *vertical* Poiseuille flow will not experience any sideways force, and so should not migrate. This will be true even in the presence of a buoyancy force. Hence the dynamical reason for the observations of rapid radial migration (Jeffrey & Pearson 1965) must be found in non-linear effects.

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